EXERCISE – IV
HINTS & SOLUTIONS

Sol.1 $(1+x)^n = C_0 + C_1x^1 + C_2x^2 + \ldots C_rx^r \ldots + C_nx^n \ldots$(i)
$(x+1x)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_nx^0 \ldots$(ii)
Multiply (i) & (ii) and compair the coeff. of x^n both side $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

$= \text{coeff. of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n = \dfrac{(2n)!}{(n!)(n!)}$

Sol.2 multiply (i) & (ii) and compair the coeff. of x^{n+1} or x^{n-1} both side $C_0C_1 + C_1C_2 + \ldots + C_{n-1}C_n$
$= \text{coeff. of } x^{n+1} \text{ or } x^{n-1} \text{ in } (1+x)^{2n} = {}^{2n}C_{n+1} = {}^{2n}C_{n-1}$

$= \dfrac{(2n)!}{(n+1)!(n-1)!}$

Sol.3 Let $S = 0.C_0 + 1C_1 + 2C_2 + 3C_3 + \ldots + nC_n$
$S = n.C_0 + (n-1)C_1 + (n-2)C_2 + \ldots 1.C_{n-1} + 0C_n$
$2S = n(C_0 + C_1 + C_2 + \ldots + C_n)$
$\Rightarrow 2S = n \cdot 2^n \Rightarrow S = n \cdot 2^{n-1}$

Aliter :

$$\sum T_p = \sum_{p=1}^{n} P.\,{}^nC_p = \sum_{1}^{n} P.\frac{n}{p}\,{}^{n-1}C_{p-1} = n\sum_{1}^{n}{}^{n-1}C_{p-1}$$

$S = n[{}^{n-1}C_0 + {}^{n-1}C_1 + \ldots + {}^{n-1}C_{n-1}] = n.2^{n-1}$

Aliter :
$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$
diff. w.r.t to x
$n(1+x)^{n-1} = C_1 + 2C_2x + 3.C_3x^2 + \ldots + nC_nx^{n-1}$
Put $x = 1$
$C_1 + 2C_2 + 3 \cdot C_3 + \ldots + nC_n = n(1+1)^{n-1}$
$= n \cdot 2^{n-1}$

Sol.4 $S = C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n$
$S = (n+1)C_0 + (n)C_1 + (n-1)C_2 + \ldots + 1C_n$
$\Rightarrow 2S = (n+2)[C_0 + C_1 + C_2 + \ldots + C_n] = (n+2)2^n$
$\Rightarrow S = (n+2).2^{n-1}$

Aliter : $\displaystyle\sum T_{P+1} = \sum_{P=0}^{n}(P+1)C_p = \Sigma PC_P + \Sigma C_p$

$= n \cdot 2^{n+1} + 2^n = 2^{n-1}[n+2]$

Sol.5 $S = C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n$
$S = (2n+1)C_0 + (2n-1)C_1 + \ldots + C_n$
$2S = (2n+2)[C_0 + C_1 + \ldots C_n] = 2(n+1)2^n$
$\Rightarrow S = (n+1)2^n$

Aliter :

$$T_{p+1} = \sum_{P=0}^{n}(2P+1)C_p = 2\Sigma P \cdot C_p + \Sigma C_p$$

$= 2.n.2^{n-1} + 2^n = 2^n(n+1)$

Sol.6 L.H.S. $= {}^{n+1}C_1 \cdot {}^{n+1}C_2\,{}^{n+1}C_3 \ldots {}^{n+1}C_n$

$$= \frac{(n+1)}{1}C_0 \cdot \frac{(n+1)}{2}C_1 \cdot \frac{(n+1)}{3}C_2 - \ldots \frac{(n+1)}{n}C_{n-1}$$

$$= \frac{C_0.C_1.C_2\ldots C_{n-1}}{n!}(n+1)^n = \text{R.H.S.}$$

Sol.7 Given $P_n = C_0 \cdot C_1 \cdot C_2 \ldots C_{n-1}C_n$

$$\therefore \frac{P_{n+1}}{P_n} = \frac{{}^{n+1}C_0\,{}^{n+1}C_1\,{}^{n+1}C_2\ldots{}^{n+1}C_n\,{}^{n+1}C_{n+1}}{{}^nC_0\,{}^nC_1\,{}^nC_2\ldots{}^nC_{n-1}\,{}^nC_n}$$

$\{{}^{n+1}C_{n+1} = 1 = {}^nC_n;\ {}^{n+1}C_0 = {}^nC_0 = 1$

$$\Rightarrow \frac{P_{n+1}}{P_n} = \frac{\frac{(n+1)}{1}C_0.\frac{(n+1)}{2}.C_1\ldots\frac{(n+1)}{n}C_{n-1}}{C_0.C_1.C_2\ldots C_{n-1}}$$

$$\Rightarrow \frac{P_{n+1}}{P_n} = \frac{(n+1)^n}{n!}$$

Sol.8 L.H.S. $= \left(\dfrac{n-0}{1}\right) + 2\left(\dfrac{n-1}{2}\right) + 3\left(\dfrac{n-2}{3}\right) + \ldots + n\left(\dfrac{n-(n-1)}{n}\right)$

$= n + (n-1) + (n-2) + \ldots + [n-(n-1)]$

$= n^2 - [1 + 2 + 3 + \ldots + (n-1)] = n^2 - \dfrac{n(n-1)}{2}$

$= \dfrac{2n^2 - n^2 + n}{2} = \dfrac{n(n+1)}{2} = \text{R.H.S}$

Sol.9 L.H.S. $= C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots + \dfrac{C_n}{(n+1)}$

$$\sum T_{P+1} = \sum_{p=0}^{n}\frac{C_p}{P+1} = \sum_{p=0}^{n}\frac{{}^nC_p}{P+1}\times\frac{(n+1)}{(n+1)}$$

$$= \frac{1}{(n+1)}\sum_{0}^{n}\frac{n+1}{p+1}\cdot{}^nC_p = \frac{1}{(n+1)}\sum_{0}^{n}{}^{n+1}C_{p+1}$$

$$= \frac{1}{(n+1)} [{}^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_{n+1}]$$

$$= \frac{1}{(n+1)} [2^{n+1} - {}^{n+1}C_0] = \frac{1}{(n+1)} (2^{n+1} - 1) = \text{R.H.S.}$$

Aliter :

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Integration w.r. to x

$$\frac{(1+x)^{n+1}}{n+1} + k = C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots + C_n\frac{x^{n+1}}{n+1}$$

where k is integral coeff.

If $x = 0 \Rightarrow k = -\frac{1}{(n+1)}$

$$\frac{(1+x)^{n+1}}{(n+1)} - \frac{1}{(n+1)} = C_0x + \frac{C_1}{2}x^2 + \frac{C_2}{3}x^3 + \ldots + \frac{C_n}{n+1}x^{n+1} \quad \ldots\text{(iii)}$$

Put x = 1

$$\frac{(2^{n+1}-1)}{(n+1)} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots + \frac{C_n}{n+1}$$

Sol.10 L.H.S. $= 2.C_0 + \frac{2^2C_1}{2} + \frac{2^3C_2}{3} + \frac{2^4C_3}{4} + \ldots + \frac{2^{n+1}C_n}{n+1}$

$$\sum T_{P+1} = \sum_0^n \frac{2^{P+1}C_p}{P+1} = \sum_0^n \frac{1}{(p+1)}.\frac{(n+1)}{(n+1)}\ {}^nC_P\, 2^{P+1}$$

$$= \frac{1}{(n+1)} \sum_{p=0}^{n} {}^{n+1}C_{p+1}\, 2^{P+1}$$

$$= \frac{1}{(n+1)} [(1+2)^{n+1} - {}^{n+1}C_0\, 2^0]$$

$$= \left(\frac{3^{n+1}-1}{n+1}\right) = \text{R.H.S.} \left\{\because \sum_0^n {}^nC_rx^r = (1+x)^r\right\}$$

Sol.11 By Question 1 (i) & (ii)

Multiply (i) & (ii) and compair of coeff. of x^{n-r} or x^{n+r}

$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n+r}C_n = x^{n+r}$ in $(1 + x)^{2n}$

$$= {}^{2n}C_{n+r} = {}^{2n}C_{n-r} = \frac{(2n)!}{(n-r)!(n+r)!}$$

Sol.12 L.H.S. $= C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \frac{C_n}{n+1}$

$$\sum T_{p+1} = \sum_0^n \frac{(-1)^pC_p}{p+1}$$

$$= \frac{1}{(n+1)} \sum_0^n {}^{n+1}C_{p+1}\, (-1)^p$$

$$= \frac{1}{(n+1)} [{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - {}^{n+1}C_4 \ldots + (-1)^n\, {}^{n+1}C_{n+1}]$$

$$= \frac{1}{(n+1)} [{}^{n+1}C_0 - ({}^{n+1}C_0 - {}^{n+1}C_1 + {}^{n+1}C_2 + \ldots - (-1)^n\, {}^{n+1}C_{n+1})]$$

$$= \frac{1}{(n+1)} [1 - 0] = \frac{1}{(n+1)} = \text{R.H.S.}$$

Sol.13 L.H.S. $= C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n + 1) C_n$

$$= \sum_{P=0}^{n} T_{P+1} \sum_{P=0}^{n} (-1)^P (P+1)\, C_P$$

$$= \sum_{P=0}^{n} (-1)^P\, P\, .\, C_P + \sum_{P=0}^{n} (-1)^p.C_p$$

$$\sum_{P=0}^{n} (-1)^p p.\frac{n}{p}\, {}^{n-1}C_{p-1} + \sum_{P=0}^{n} {}^nC_p(-1)^p$$

$$= (-n) \sum_{P=1}^{n} {}^{n-1}C_{p-1}(-1)^{p-1} + \sum_{P=0}^{n} {}^nC_p(-1)^p$$

$= (-n)\, [1 - 1]^{n-1} + [1 - 1]^n = 0 + 0 = 0 =$ R.H.S.

Sol.14 L.H.S. $= (n-1)^2 .C_1 + (n-3)^2.C_3 + (n-5)^2 . C_5 + \ldots$

$$= \sum_{r=1}^{n} (n-r)^2.C_r = \sum_1^n (n^2 - 2nr + r^2)\, C_r$$

$$= n^2 \sum_1^n C_r - 2n \sum_1^n r.C_r + \sum_1^n r^2.C_r$$

$= n^2 (2^n - 1) + (1 - 2n)\, n \,.\, 2^{n-1} + n(n-1)\, 2^{n-2}$
$= n^2\, 2^n - n^2 + n.2^{n-1} - n^2 2^n + n(n-1) 2^{n-2}$
$= -n^2 + n(n+1)\, 2^{n-2}$ ....(1)

Again L.H.S. $= \sum_{r=1}^{n} (n-r)^2 C_r\, (-1)^{r+1}$

$= \sum_{r=1}^{n} (n^2 - 2nr + r^2) C_r (-1)^{r+1}$

$= n^2 \sum_{r=1}^{n} {}^nC_r (-1)^{r+1} + (1-2n) \sum_{1}^{n} r.C_r(-1)^{r+1}$

$+ n(n-1) \sum_{1}^{n} {}^{n-2}C_{r-2}(-1)^{r+1}$

$= -n^2 [(1-1)^n - 1] + (1-2n)n\, [1-1]^{n-1}$
$- n(n-1)\, [1-1]^{n-2}$

$= -n^2 [0-1] + 0 + 0 = n^2$ ....(2)

Adding (1) & (2)

$2[(n-1)^2 . C_1 + (n-3)^2 . C_3 + (n-5)^2 C_5 + ...]$
$= -n^2 + n(n+1)\, 2^{n-2} + n^2$

$\Rightarrow (n-1)^2 . C_1 + (n-3)^2 . C_3 + (n-5)^2 C_5 + ...$

$= \frac{n(n+1)2^{n-2}}{2} = n(n+1)\, 2^{n-3}$

Sol.15 L.H.S. $= 1.C_0^{\,2} + 3.\,C_1^{\,2} + 5.C_2^{\,2} + ... + (2n+1)\, C_n^{\,2}$

Let $S = 1.\, C_0^{\,2} + 3.C_1^{\,2} + 5.C_2^{\,2} + ... + (2n+1)\,.\, C_n^{\,2}$

$S = (2n+1)\, C_0^{\,2} + (2n-1)\, C_1^{\,2} + (2n-2)\, C_2^{\,2} + ... + 1\,.\, C_n^{\,2}$

Adding

$2S = (2n+2)\, [C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + ... + C_n^{\,2}]$

$2S = 2(n+1)\, {}^{2n}C_n$

$S = (n+1)\, {}^{2n}C_n = \frac{(n+1)(2n)!}{n!.n!}$ = R.H.S.

Sol.16 $(1 + x + x^2)^n$

$= a_0 + a_1 + a_2 x + ... + a_n x^n + ... + a_{2n} x^{2n}$ ...(i)

Replace $x \to \left(-\frac{1}{x}\right)$

$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - .... + \frac{a_2 x}{x^{2n}}$

$\Rightarrow \frac{(x^2 - x + 1)^n}{x^{2n}}$

$= \frac{a_0 x^{2n} - a_1 x^{2n-1} + a_2 x^{2n-1} + ... + a_{2n}}{x^{2n}}$ ....(ii)

Multipy (i) & (ii) then L.H.S are

$\{(x^2+1)^2 - x^2\}^n$

$= \left(a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2\right) x^{2n} + (....)$

$(1 + x^2 + x^4)^n$

$= \left(a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2\right) x^{2n} + (...)$ ...(iii)

Put x = 1 in (i)

$\Rightarrow a_0 + a_1 + a_2 + a_3 ... + a_{2n} = (1+1+1)^n = 3^n$ ....(iv)

Put x = –1 in (i)

$\Rightarrow a_0 - a_1 + a_2 - a_3 ... + a_{2n} = (1 - 1 + 1)^3 = 1^n = 1$ .....(v)

add (iv) & (v)

$\Rightarrow a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n + 1}{2}$ ....(vi)

Subtract (iv) – (v)

$\Rightarrow a_1 + a_3 + a_5 + ...\ a_{2n-1} = \frac{3^n - 1}{2}$ ....(vii)

(i) L.H.S. = $a_0 a_1 - a_1 a_2 + a_2 a_3 - .....$

Multiply (i) & (ii) and compair of x^1 or $(-x^{-1})$

$a_0 a_1 - a_1 a_2 + a_2 a_3 .... = x^1$ in $\frac{(1 + x^2 + x^4)^n}{x^{2n}}$

$= x^{2n+1}$ in $(1 + x^2 + x^4)^n$

$= 0$ [$\because$ all degree of in $(1 + x^2 + x^4)^n$ is even]

(ii) L.H.S. = $a_0 a_2 - a_1 a_3 + a_2 a_4 ... + a_{2n-2}\, a_{2n}$

Multiply (i) & (ii) and compair of x^2 or x^{-2}

$a_0 a_2 - a_1 a_3 + a_2\, a_4 - ... + a_{2n-2}\, a_{2n}$

$= x^2$ in $\frac{(1 + x^2 + x^4)^n}{x^{2n}} = x^{2n+2}$ in $(1 + x^2 + x^4)^n$

$= a_{n+1}$

or $= x^{-2}$ in $\frac{1}{\frac{(1 + x^2 + x^4)}{x^{2n}}} = x^{2n-2}$ in $(1 + x^2 + x^4)^n$

$= a_{n-1}$

(iii) Put x = ω in equation (i)

$(1 + \omega + \omega^2)^n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + ...$
$+ a_{2n}\omega^{2n}$

$\because$ ω is cube root of unity

$\therefore$ $\omega^3 = 1,\ 1 + \omega + \omega^2 = 0,$

$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i,\ \omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$\Rightarrow 0^n = (a_0 + a_3 + a_6 + ...)\omega^3 + (a_1 + a_4 + a_7 + ...)\, \omega$
$+ (a_2 + a_5 + a_8 + ...)\omega^2$

$\Rightarrow E_1 + E_2\omega + E_3\omega^2 = 0$

$\therefore E_1 - \frac{E_2}{2} - \frac{E_3}{2} = 0$ & $E_2 - E_3 = 0 \Rightarrow E_2 = E_3$

$\Rightarrow E_1 = E_2 \quad \therefore E_1 = E_2 = E_3$

Put x = 1 in (i)

$a + a_1 + a_2 + a_3 + \dots a_{2n} = (1 + 1 + 1)^n = 3^n$

$\Rightarrow (a_0 + a_3 + a_6 + \dots) + (a_1 + a_4 + a_7 + \dots) + (a_2 + a_5 + a_8 + \dots) = 3^n$

$\Rightarrow E_1 + E_2 + E_3 = 3^n \Rightarrow 3E_1 = 3^n$

$\therefore E_1 = 3^{n-1} = E_2 = E_3$

Sol.17 $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ ...(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n$ ...(ii)

Multiply (i) & (ii) and compair coeff. of x^{n+2} or x^{n-2}

$C_0C_2 + C_1C_3 + C_2C_4 + \dots + C_{n-2}C_n$

$= x^{n+2}$ or x^{n-2} in $(1 + x)^{2n}$

$$= {}^{2n}C_{n-2} = \frac{(2n)!}{(n-2)!(n+2)!}$$

Sol.18 $(1 + x)^n = C_0 + C_1x^1 + C_2x^2 + \dots + C_nx^n$

$$\sum_{0\le i}\sum_{<j\le n} C_iC_j = 2^{2n-1} - \frac{(2n)!}{2(n!)^2}$$

$$\sum_{r=1}^{n}(C_0C_r + C_rC_{r+1} + C_2C_{r+2} + \dots + C_{n-r}\,.\,C_n) = \sum_{r=1}^{n} {}^{2n}C_{n+r}$$

$$\sum_{r=1}^{n} {}^{2n}C_{n+r} = {}^{2n}C_{n+1} + {}^{2n}C_{n+2} + {}^{2n}C_{n+3} + \dots + {}^{2n}C_{2n}$$

$$\Rightarrow \sum_{r=1}^{n} {}^{2n}C_{n+1} \ ({}^{2n}C_0 + {}^{2n}C_1 + \dots + {}^{2n}C_n)$$

$$= ({}^{2n}C_0 + {}^{2n}C_1 + \dots + {}^{2n}C_n) + {}^{2n}C_{n+1} + {}^{2n}C_{n+2} + \dots + {}^{2n}C_{2n}$$

$$= \left(\sum_{r=1}^{n} {}^{2n}C_{n+r}\right) + ({}^{2n}C_{n+1} + {}^{2n}C_{n+2} + {}^{2n}C_{n+3} + \dots + {}^{2n}C_{2n})$$

$+ {}^{2n}C_n = 2^{2n}$

$$\Rightarrow \left(\sum_{r=1}^{n} {}^{2n}C_{n+r}\right) + \left(\sum_{r=1}^{n} {}^{2n}C_{n+r}\right) + {}^{2n}C_n = 2^{2n}$$

$$\Rightarrow \sum_{r=1}^{n} {}^{2n}C_{n+r} = \frac{2^{2n} - {}^{2n}C_n}{2} = 2^{2n-1} - \frac{(2n)!}{2.(n!)^2}$$

Sol.19 $\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + \dots + \sqrt{C_n} \le 2^{n-1} + \frac{n-1}{2}$

Multiply 2 both side

$\Rightarrow 2\sqrt{C_1} + 2\sqrt{C_2} + 2\sqrt{C_3} + \dots + 2\sqrt{C_n} \le 2^n + (n - 1)$

$\Rightarrow 2\sqrt{C_1} + 2\sqrt{C_2} + 2\sqrt{C_3} + \dots + \sqrt{C_n} \le C_0 + C_1 + C_2 + C_3 + \dots + C_n + n - 1$

$\{\because \ 2^n = C_0 + C_1 + C_2 + \dots + C_n$ & $C_0 = 1\}$

$\Rightarrow 2\sqrt{C_1} + 2\sqrt{C_2} + 2\sqrt{C_3} + \dots\dots\dots + 2\sqrt{C_n}$

$\le C_1 + C_2 + C_3 + .. C_n + (1 + 1 + 1 + \dots n \text{ times})$

$\Rightarrow 2\sqrt{C_1} + 2\sqrt{C_2} + 2\sqrt{C_3} + \dots + 2\sqrt{C_n}$

$\le (C_1 + 1) + (C_2 + 1) + (C_3 + 1) + \dots + (C_{n+1})$

$\because$ A.M. $\ge$ G.M. $\Rightarrow \frac{C_1 + 1}{2} \ge \sqrt{C_1 \times 1}$

$\Rightarrow C_1 + 1 \ge 2\sqrt{C_1}$

$\therefore 2\sqrt{C_1} + 2\sqrt{C_2} + 2\sqrt{C_3} + \dots + 2\sqrt{C_n}$

$\le (C_1 + 1) + (C_2 + 1) + \dots + (C_n + 1)$ H.P.

Sol.20 $\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + \dots + \sqrt{C_n}$

$\le [n(2^n - 1)]^{1/2}$ for $n \ge 2$

$\{\because$ Root mean square $\ge$ A.M.$\}$

$$\Rightarrow \sqrt{\frac{(\sqrt{C_1})^2 + (\sqrt{C_2})^2 + \dots + (\sqrt{C_n})^2}{n}}$$

$$\ge \frac{\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + \dots + \sqrt{C_n}}{n}$$

$$\Rightarrow \frac{\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + \dots + \sqrt{C_n}}{n}$$

$$\le \sqrt{\frac{C_1 + C_2 + C_3 + \dots + C_n}{n}}$$

$$\Rightarrow \sqrt{C_1} + \sqrt{C_2} + \dots + \sqrt{C_n} \le \sqrt{n(C_1 + C_2 + \dots + C_n)}$$

$\Rightarrow \sqrt{C_1} + \sqrt{C_2} + \dots + \sqrt{C_n}$

$\le \sqrt{n(C_0 + C_1 + C_2 + \dots + C_n - C_0)}$

$\Rightarrow \sqrt{C_1} + \sqrt{C_2} + \dots + \sqrt{C_n} \le [n(2^n - 1)]^{1/2}$